§ 12.3

Aim: Recall: (12.11.5):

$$\prod_{j\geq 1}(1-q^j)^{\dim \mathfrak{g}_j(s)} = \sum_{w\in W^s} \varepsilon(w) D_s(w(\rho)) q^{\langle \rho - w(\rho), h^s\rangle}$$

· Macdonald's η-function identities:

$$\eta^{\dim \mathfrak{g}} = \sum_{\alpha\in M} d(h^\vee \alpha)\, q^{|\bar\rho + h^\vee \alpha|^2 / 2h^\vee}$$

· let $\mathfrak{g}$ is a simple f.d. Lie alg of type X_N, $\mu \in \mathrm{Aut}(S(A))$ order$(\mu) = r$. $\mathfrak{g} = \bigoplus \mathfrak{g}_{\bar j}$ $\mathbb{Z}/r\mathbb{Z}$-gradation

· Given a nonzero sequence of integer $s = (s_1, \dots, s_l)$ where $s_i \geq 0$ set $m = r\sum a_i s_i$, and $\sigma_{s,r} \in \mathrm{Aut}(\mathfrak{g})$ s.t.

$$\sigma_{s,r}(E_j) = \varepsilon^{s_j} E_j, \quad \sigma_{s,r}(F_j) = \varepsilon^{-s_j} F_j, \quad \sigma_{s,r}(H_j) = H_j$$

$$\varepsilon = e^{\frac{2\pi}{m} i}$$

let $\mathfrak{g} = \bigoplus_{\bar j} \mathfrak{g}_{\bar j}(s;r)$ be the $\mathbb{Z}/m\mathbb{Z}$ gradation

put $d_{\bar j}(s;r) = \dim \mathfrak{g}_{\bar j}(s;r)$, $(j\in\mathbb{Z})$ $\bar j \in \mathbb{Z}/m\mathbb{Z}$.

· Lemma 12.3. let $\mathfrak{g}(A)$ be the affine algebra of type $X_N^{(r)}$ and let $\mathfrak{g}(A) = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j(s)$ be its $\mathbb{Z}$-gradation of type s

Then $\dim \mathfrak{g}_j(s) = d_{\bar j}(s;r)$

· The left hand side of (12.11.5) is

$$R_{\text{Left}} = \prod_{j\geq 1}(1-q^j)^{d_{\bar j}(s;r)}$$

· write R_{Right} in terms of $\mathring{\mathfrak{g}}$

· let $\Delta_s = \sum k_i \alpha_i$ where $\alpha_i \in \mathring{\Delta}$ s.t. $s_i = 0$

W_s is generated by r_α, $\alpha \in \Delta_s$

$W = \underbrace{W_s T}\, \mathring{W}^s$, set $T\mathring{W}^s = W^s$ (representation)

$$= \{ t_\alpha w \mid \alpha \in M,\ w \in \mathring{W}^s \}$$

· Claim: (12.3.3) $t_\alpha w(\rho) = h^\vee\Lambda_0 + w(\bar\rho) + h^\vee\alpha - \left(\frac{h^\vee}{2}|\alpha|^2 + (w(\bar\rho)|\alpha)\right)\delta$

· Hence we have $\langle \alpha_i, h^s\rangle = s_i$ $(i=0,\dots,n)$ $h^s\in H$

$\langle \rho - t_\alpha w(\rho), h^s\rangle = \langle \bar\rho + \cancel{h^\vee\Lambda_0}, h^s\rangle - \langle w(\bar\rho), h^s\rangle - \langle h^\vee\alpha, h^s\rangle$

(※) $+ \left(\frac{h^\vee}{2}|\alpha|^2 + (w(\bar\rho)|\alpha)\right)\langle\delta, h^s\rangle$

$= \langle\bar\rho, h^s\rangle - \langle w(\bar\rho), h^s\rangle - h^\vee\langle\alpha, h^s\rangle + \frac{m}{r}\left(\frac{h^\vee}{2}|\alpha|^2 + (w(\bar\rho)|\alpha)\right)$

· we define $\gamma_s \in \mathring{H}^*$ by

(12.3.2) $(\gamma_s|\alpha_i) = \frac{r s_i}{m}$ $(i=1,\dots,l)$

Then (※) can be written as follows

$\boxed{(12.3.4)}$ $\langle \rho - t_\alpha w(\rho), h^s\rangle = \frac{m}{2h^\vee r}\left(|h^\vee\alpha + w(\bar\rho) - h^\vee\gamma_s|^2 - |\bar\rho - h^\vee\gamma_s|^2\right)$

· Using (12.3.1) and (12.3.4) we deduce from (11.11.5)

(12.3.5) $q^{\frac{m}{2h^\vee r}|\bar\rho - h^\vee\gamma_s|^2}\prod_{j\ge1}(1-q^j)^{d_j(s;r)}$

$= \sum_{w\in\mathring{W}^s}\varepsilon(w)\sum_{\alpha\in M} D_s(\quad)\, q^{(\quad)}$

· The η-function identity is a special case of (12.3.5) for

$s=(1,0,0,\dots,0)$, $r=1$

Then $m = a_0 = 1$, moreover $(\gamma_s|\alpha_i) = 0$ $(i=1,\dots,l)$

$|\bar\rho - h^\vee\gamma_s|^2 = |\bar\rho|^2$

$|w(\bar\rho) + h^\vee\alpha - h^\vee\gamma_s|^2 = |w(\bar\rho) + h^\vee\alpha|^2$

Since $\Delta_s = \mathring{\Delta}$, then $\mathring{W}^s = 1$, i.e. $\varepsilon(w)=1$, $w(\bar\rho)=\bar\rho$

& $\dim\mathfrak{g}_j(s,r) = \dim\mathring{\mathfrak{g}} = \frac{12|\bar\rho|^2}{h^\vee}$, $\mathring{\Delta}^\vee_{s+} = \mathring{\Delta}^\vee_+$, $\rho_s = \bar\rho$

then (12.3.5) $q^{\frac{|\bar\rho|^2}{2h^\vee}}\prod_{j\ge1}(1-q^j)^{\dim\mathring{\mathfrak{g}}} = \sum_{\alpha\in M}\prod_{\alpha\in\mathring{\Delta}^\vee_+}\frac{\langle\bar\rho + h^\vee\alpha, \alpha\rangle}{\langle\bar\rho, \alpha\rangle}\, q^{\frac{|\bar\rho+h^\vee\alpha|^2}{2h^\vee}}$

§12.4.

Recall $K = \sum_{i=0}^{l} a_i^\vee \alpha_i^\vee$.

$K|_{L(\Lambda)} = \langle \Lambda, K \rangle 1_{L(\Lambda)}$

- $\langle \Lambda, K \rangle = \langle \lambda, K \rangle$, with $\forall \lambda \in P(\Lambda)$
- The number (12.4.1) $k := \langle \Lambda, K \rangle = \sum_{i=0}^{l} a_i^\vee \langle \Lambda, \alpha_i^\vee \rangle$ is called the level of $\Lambda \in H^*$ or of the module $L(\Lambda)$
- if $\Lambda \in P_+$, then $k \geq 0$ & $k \in \mathbb{Z}$, $k = 0$ iff $\langle \Lambda, \alpha_i^\vee \rangle = 0$, $\forall i$
- Hence by (9.10.1). ($\dim L(\Lambda) = 1$ iff $\Lambda|_{H} = 0$)

…

- (12.4.2) $\langle \rho, K \rangle = \sum_{i=0}^{l} a_i^\vee = h^\vee$
- Let Λ_i ($i = 0, \ldots, l$) be the fundamental weights

$\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$, $j = 0, \ldots, l$, and $\langle \Lambda_i, d \rangle = 0$ $\quad$ $\underset{\sim\sim\sim}{\langle \delta, d \rangle = 1}$

Note that (12.4.3) $\Lambda_i = \bar{\Lambda}_i + a_i^\vee \Lambda_0$, where $\bar{\Lambda}_0 = 0$, and $\bar{\Lambda}_1, \ldots, \bar{\Lambda}_l$ are the fundamental weights of $\mathring{\mathfrak{g}}$

- (12.4.4) $P_+ = \sum_{i=0}^{l} \mathbb{Z}_+ \Lambda_i + \mathbb{C}\delta$
- Claim. $\text{level}(\Lambda) = 1$, $\Lambda \in P_+$, iff $\Lambda \equiv \Lambda_i \mod \mathbb{C}\delta$ and $a_i^\vee = 1$

$\Lambda = \sum k_i \Lambda_i + c\delta$

$\Rightarrow \langle \Lambda, K \rangle = \sum k_i a_i^\vee = 1$

$\Rightarrow$ for some i, $k_i = 1$, $a_i^\vee = 1$

$\Leftarrow$ ✓

Claim. If A is symmetric or $r \geq 1$.

(12.4.5) $\text{level}(\Lambda_i) = 1$ iff $i \in \text{Aut}(S(A)) \cdot 0$

- Lem 12.4: if $\text{Re}\langle \lambda, K \rangle > 0$, then

$\left| (W \cdot \lambda) \cap \{\mu \in H^* \mid \text{Re}\langle \mu, \alpha_i^\vee \rangle \geq 0 \text{ for all } i\} \right| = 1$

In part. if $\text{level}(\lambda) > 0$, $\lambda \in P$, then

$|P_+ \cap W \cdot \lambda| = 1$